Characterization of Anti Q-fuzzy R-Sub modules over Commutative Rings

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ABSTRACT
We introduce the concept of anti Q-fuzzy R- sub modules over a commutative ring with respect to t- norm. Some Properties of anti Q- fuzzy R- sub modules are investigated. In Particular, we consider properties of intersection and direct product for anti Q- fuzzy R- sub modules.

Key words: Anti Q-fuzzy R-sub modules, Direct product, T-norm, sensible.

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INTRODUCTION
The concept of fuzzy set [7] was applied to group theory in [4]. Further, the concept of fuzzy subgroup was generalized to the case where in the definition of an algebraic structure one uses a t- norm [1] instead of the operation min. A significant number of publications are devoted to research of properties of fuzzy Abelian groups (see, for example, [2] and fuzzy rings [3].The concept of Q- fuzzy submodules was generalized by [6]. In this paper we introduce the concept of anti Q- fuzzy R- sub module and investigate some properties of anti Q- fuzzy R- sub modules. To guarantee a sufficient generality of exposition and account for a variety of various applications, all concepts used here are considered with respect to t-norms.

Preliminaries

Definition 2.1 : Let $\mu_{\tilde{A}} : U \rightarrow [0,1]$ be any function and A be a crisp set in the universe 'U' then the ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x) / x \in U\}$ is called a fuzzy set and $\mu_{\tilde{A}}$ is called a membership function.

Definition 2.2: By a t- norm 'T' , we mean a function T: $[0,1]\times [0,1]\rightarrow [0,1]$ satisfying the following conditions ;

(T1) $T(0 ,x) = 0$

(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$

(T3) $T(x,y) = T(y,x)$

(T4) $T(x, T(y,z)) = T(T(x,y),z)$, for all $x,y,z \in [0,1]$.

Proposition 2.3: For a t-norm , then the following statement holds $T(x,y) \leq \min\{x,y\}$, for all $x,y \in [0,1]$.

Definition 2.4: Let 'T' be a t-norm. A fuzzy set 'A' in 'R' is said to be sensible with respect to 'T' if Im(A) c ΔT, where

$\Delta T = \{ T(\alpha, \alpha) = \alpha / \alpha \in [0,1] \}$.

Definition 2.5 : Let Q and G be a set and a group respectively. A mapping $\mu: G \times Q \rightarrow [0,1]$ is called a Q – fuzzy set.

Definition 2.6: Let M be a module over a commutative ring R.. A function $A : M\times Q \rightarrow [0,1]$ is called anti Q-fuzzy R- sub module of 'M' with respect to s-norm S if and only if for all $x,y \in M$, $q \in Q$ and for any $\alpha \in R$.

(AFM1) $A(x+y,q) \geq T \{ A(x,q) , A(y,q) \}$, for all $x,y \in A$.

(AFM2) $A(\alpha x,q) \geq A(x,q)$ for all $\alpha,x \in R$.

(AFM3) $A(0,q) = 1$

Definition 2.7 : Define

T_n $((x_1,x_2, \ldots, x_n),q)$ = $T(x_i, T_{n-1}((x_1,x_2,\ldots x_{i-1}, x_{i+1},\ldots x_n),q))$ for all $1 \leq i \leq n$, $n \geq 2$, $T_1 = T$. Also define $T_\infty(x_1,x_2,\ldots, q) = \lim T_n((x_1,x_2,\ldots,x_n),q)$ as $n \rightarrow \infty$.

Definition 2.8: By the intersection of fuzzy subsets A_1 and A_2 in a set X with respect to an t-norm T we mean the fuzzy subset $A = A_1 \cap A_2$ in the set X such that for any x ε X $A(x,q) = (A_1 \cap A_2)(x,q) = T(A_1(x,q), A_2(x,q))$. By the intersection of a collection of fuzzy subsets $\{A_1,A_2,\ldots\}$ in a set X with respect to a t- norm T we mean the fuzzy subset $\cap A_i$ such that for any x ε A, $(\cap A_i)(x,q) = T_\infty\{A_1(x,q), A_2(x,q),\ldots\}$.

Definition 2.9: By the direct product of fuzzy sets $\{A_1,A_2,\ldots\}$ with respect to t- norm T we Mean the fuzzy subset $A = \prod A_i$ such that $A\{(x_1,x_2,\ldots x_n),q\}$

$= (\prod A_i)\{(x_1,x_2,\ldots x_n),q\}$

$= T_n\{A_1(x_1,q), A_2(x_2,q) \ldots A_n(x_n,q)\}$.

Properties of Anti Q- fuzzy R-sub modules

Proposition 3.1: If R is a ring with identity and a t-norm T for all x ε [0,1] satisfies the condition$T(x,x)=x$. (T5)

Then condition (AFM2) in Definition (2.6) for any αεR is equivalent to the condition

$A(\alpha x,q)=A(x,q)$. (AFM2[1])

Proof: Let condition (AFM1) and (AFM2) be fulfilled and 1 be the identity element in the ring R. Then

$A(x,q) = A(\alpha x + (1-\alpha x),q) \geq T(A(\alpha x,q), A(1-\alpha x),q) \geq T A(\alpha x,q), T(A(x,q), A(-\alpha x,q)))) \geq T(A(x,q), T(A(x,q), A(-\alpha x,q)))$. Taking into consideration conditions (T2) and (T5) for the t-norm T and again applying AFM2), we obtain $T(A(x,q), T(A(x,q), A(-\alpha x,q))) = T(T(A(x,q), A(x,q),A(-\alpha x,q)) = T(A(x,q), A(-\alpha x,q)) \geq T(A(x,q), A(\alpha x,q))$. Thus we have $A(x) = T(A(x,q), A(x,q)) \geq T(A(x,q), A(\alpha x,q))$.From here, using condition (T4), we conclude that $A(x,q) \geq A(\alpha x,q)$ (1) From (1) and condition (AFM2) we obtain (AFM2[1]).

Proposition 3.2 : If R is a field , then condition (AFM2) in Definition 2.6 for any αε R, $\alpha \neq 0$, is equivalent to condition (AFM2[1]).

Proof: Let the condition (AFM2) be satisfied. If $\alpha \neq 0$, then applying (AFM2) we obtain $A(\alpha x,q) \geq A(x,q) = A(1/\alpha(\alpha x,q)) \geq A(\alpha x,q)$. From here it follows that $A(\alpha x,q) = A(x,q)$, i.e., condition (AFM2[1]) is satisfied.

Proposition 3.3: If a fuzzy set A is normal, then condition(AFM3) in Definition 2.6 follows from conditions (AFM1) and (AFM2).

Proof: Let, for a fuzzy set A, condition (AFM1) and (AFM2) be fulfilled. Since the fuzzy set A is normal, there exists x ε M such that (1) is fulfilled. Then, for such x, applying conditions (AFM1) and (AFM2), we obtain $A(0,q) = A(x-x,q) \geq T(A(x,q), A(-x,q)) = T(1, A(-x,q) = A(-x,q) \geq A(x,q) = 1$. From here it follows that $A(0,q) = 1$. Thus, condition (AFM2[1]) is satisfied.

Proposition 3.4 : If A is anti Q- fuzzy R- sub module of M with respect to a t- norm T, then $M_1 = \{ x / x \in M, A(x,q) = 0\}$ is a sub module of the module M and A is anti Q-fuzzy R- sub module of M_1 , with respect to the t- norm T.

Proof: Let x,y ε M_1 and α ε R. Then , according to condition (AFM2), $A(x+y,q) \geq T(A(x,q), A(y,q)) = T(1,1) = 1$. Thus , $A(x+y,q) =1$. Hence x+y ε M_1. According to condition (AFM1), $A(\alpha x,q) \geq A(x,q) = 1$. Thus, we have $A(\alpha x,q) = 1$. From here it follows that αx ε M_1. Finally, according to condition (AFM3), $A(0,q) = 1$. Therefore, 0 ε M_1. Thus M_1 is a sub module of module M.

The second part of the statement of proposition 3.4 is obvious.

Proposition 3.5; If , for a unitary ring R, A is a anti Q- fuzzy R-sub module of M with respect to a t-norm T, then A is a anti Q- fuzzy subgroup of M with respect to T.

Proof: Let x ε A. Then $A(-x,q) = A((-1.x),q) \geq A(x,q)$. Thus , A is anti Q- fuzzy module of M with respect to the t- norm T.

Proposition3.6: Let 'T' be a t-norm. Then every sensible anti Q- fuzzy R- modules 'A' of R is anti Q- fuzzy R- sub modules of R.

Proof: Assume that 'A' is a sensible anti Q- fuzzy R-sub modules of R, then we have (AFM1) A(x+y,q) ≥ T (A(x,q), A(y,q)) and (AFM2) A(αx,q) ≥ A(x,q) for all x,y ε R.

Since 'A' is sensible, we have

Min { A(x,q) , A(y,q) } = T(min {A(x,q), A(y,q) , min { A(x,q), A(y,q) }

≥ T{A(x,q), A(y,q) }

≥ min { A(x,q) , A(y,q) }

and so T(A(x,q), A(y,q) } = min { A(x,q), A(y,q) }. It follows that

A(x+y,q) ≥ T(A(x,q), A(y,q) }

= min { A(x,q), A(y,q) } for all x,y in R.

Clearly A(αx,q) ≥ A(x,q) for all r,x in R. So 'A' is an anti Q-fuzzy R- sub modules of R.

Proposition 3.7: If A is anti Q-fuzzy R-sub module of M with respect to the t-norm min, then for any θ ε [0,1], M_θ = { x / x ε M, A(x,q) ≥ θ } is a sub module of the module M and A is anti Q-fuzzy R- sub module of M_θ with respect to to min.

Proof; Let x,y ε M_1, and αε R, Then A(x+y,q) ≥ min { A(x,q), A(y,q)} = min {θ,θ} = θ. Thus A(x+y,q)≥ θ. Hence x+y ε M_θ. Further, we have A(αx,q) ≥ A(x,q) ≥ θ.From here we conclude that αx ε M_θ.Finally, from A(0,q) = 1 ≥ θ it follows that 0 ε M_θ.

Proposition 3.8: Let A : B→ [0,1] be the characteristic function of a subset B is contained in M and M be an R-module. Then A is anti Q- fuzzy R-sub module of M with respect to t-norm T if and only if B is a sub module of the module M.

Proof: Let A be anti Q-fuzzy R- sub module of M with respect to T. Then, according to (AFM2) , A(αx,q) ≥ A(x,q) = 1. Hence αx ε B. Finally, according to condition (AFM3), A(0,q) = 1. Therefore, 0ε B.Thus, B is a sub module of the module M.

Conversely, Let B be a sub module of the module M. Then for any x,y ε M,

A(x+y,q) ≥ T(A(x,q), A(y,q)}.

Indeed, for any x,y ε B,

A(x+y,q) = 1 ≥ 1 = T(1,1) = T (A(x,q), A(y,q)}

For any x ε B, and y is not in B, T(A(x,q), A(y,q)) = T(1.0) = 0 ≥ A(x+y,q)

For any x is not in B, and y ε B, T(A(x,q), A(y,q)) = T(0,1) = 0 ≥ A(x+y,q)

Finally, for any x,y does not belong to B, T(A(x,q), A(y,q)) = T(0,0) = 0 ≥ A(x+y,q)

Further for all x ε M, and αε R, we have A(αx,q) ≥ A(x). Indeed, for all x ε B we have αx ε B, hence A(αx,q) = 1 ≥ A(x,q), and for all x does not belong to B we have A(x,q) = 0 ≤ A(αx,q).

Finally, since 0 ε B, we have A(0,q) = 1. Therefore, A is anti Q-fuzzy R- sub module of M with respect to T.

Proposition 3.9; The intersection of any collection of anti Q- fuzzy R- sub modules of an R-module M is anti Q-fuzzy sub module of this module.

Proof: For all x,y ε M, and any α ε R, we have

$\cap A_i$ (x+y,q) = $T_\infty(A_1(x+y,q), A_2(x+y,q), \ldots) \geq T_\infty(T_\infty(A_1(x,q), A_1(y,q)), T_\infty(A_2(x,q), A_2(y,q)), \ldots)$

$= T_\infty (T_\infty(A_1(x,q), A_2(x,q), , \ldots), T_\infty(A_1(y,q), A_2(y,q), \ldots))$

$= T_\infty ((\cap A_i)(x,q) , (\cap A_i)(y,q))$;

$(\cap A_i)(\alpha x,q) = T_\infty A_1(\alpha x,q),A_2(\alpha x,q), , \ldots) \geq T_\infty(A_1(x,q), A_2(x,q),\ldots) = (\cap A_i) (x,q)$;

$(\cap A_i)(0,q) = T_\infty(A_1(0,q), A_2(0,q), \ldots) = T_\infty(1,1,\ldots) = 1.$

Proposition is proved.

Proposition 3.10: If 'A' is a anti Q- fuzzy R-sub modules of a commutative ring R and 'θ' is an endomorphism of R, then $A_{[\theta]}$ is anti Q-fuzzy R- sub modules of R.

Proof: for any x,y ε R, we have

(i) $A_{[\theta]}(x+y,q) = A(\theta(x+y,q))$
$= A(\theta(x,q) + \theta(y,q))$

$\geq T(A_{[\theta]}(x,q), A_{[\theta]}(y,q))$

(ii) $A_{[\theta]}(\alpha x,q) = A(\theta(\alpha x,q))$

$= A(\alpha\,\theta(x,q))$

$\geq A(\theta(x,q))$

$\geq A_{[\theta]}(x,q)$, hence $A_{[\theta]}$ is anti Q-fuzzy R-sub modules of R.

Proposition 3.11: Let $\{M_1,M_2,\ldots,M_n\}$ be a collection of R-modules and $M = \prod A_i$ be its direct product. Let $\{A_1,A_2,\ldots,A_n\}$ be anti Q-fuzzy sub modules of the R-modules $\{M_1,M_2,\ldots,M_n\}$ with respect to a t-norm T.Then $A = \prod A_i$ is anti Q- fuzzy sub module of the R- module M with respect to the t-norm T.

Proof: Let x,y ε M, $x = (x_1,x_2,\ldots x_n)$, and $y = (y_1,y_2,\ldots y_n)$. Also let αε R.Then

$A(x+y,q) = A((x_1+y_1, x_2+y_2, \ldots, x_n+y_n),q) = T_n(A_1((x_1+y_1),q), A_2((x_2+y_2),q), \ldots, A_n((x_n+y_n),q))$

$\geq T_n(T(A_1(x_1,q), A_1(y_1,q)), T(A_2(x_2,q), A_2(y_2,q)), \ldots, T(A_n(x_n,q), A_n(y_n,q)))$

$= T(T_n(A_1(x_1,q), A_2(x_2,q),\ldots A_n(x_n,q)), T_n(A_1(y_1,q), A_2(y_2,q), \ldots, A(y_n,q)))$

$= T(A(x,q), A(y,q)), A(\alpha x,q)$

$= A((\alpha x_1,\alpha x_2,\ldots,\alpha x_n),q)$

$= T_n(A_1(\alpha x_1,q), A_2(\alpha x_2,q), \ldots, A_n(\alpha x_n,q))$

$\geq T_n(A_1(x_1,q), A_2(x_2,q), \ldots A_n(x_n,q))$,

$A(0,q) = A((0_1,0_2,\ldots, 0_n),q)$

$= T_n((A_1(0_1,q), A_2(0_2,q), \ldots, A_n(0_n,q))$

$= T_n(1,1,\ldots,1) = 1$. Therefore, A is anti Q-fuzzy R- sub module of the module M with respect to T.

Proposition 3.12: An onto homomorphic image of anti Q-fuzzy R- sub modules with sup property is anti Q-fuzzy R- sub modules.

Proof: Let $f: R \to R^1$ be an onto homomorphism of a ring and let 'A' be anti Q- fuzzy R-sub modules of R with sup property. Given x, y ε R, we let $x_o \,\varepsilon\, f^{-1}(x^1)$, and $y_o \,\varepsilon\, f^{-1}(y^1)$ be such that

$$A(x_o,q) = \sup_{h \varepsilon f^{-1}(x^1)} A(h,q),$$

$$A(y_o,q) = \sup_{h \varepsilon f^{-1}(y^1)} A(h,q)$$

Respectively. Then we can deduce that

$$A^f(x^1-y^1,q) = \sup_{z \varepsilon f^{-1}(x^1-y^1)} A(z,q)$$

$$\leq \min\{A(x_o,q), A(y_o,q)\}$$

$$= \min\{\sup_{h\varepsilon f^{-1}(x^1)} A(h,q), \sup_{h \varepsilon f^{-1}(y^1)} A(h,q)\}$$

$$= \min\{A^f(x^1,q), A^f(y^1,q)\}$$

$$A^f(\alpha x,q) = \sup_{z \varepsilon f^{-1}(\alpha^1 x^1)} A(z,q) \leq A(y_o,q)$$

$$= \sup_{h \varepsilon f^{-1}(y^1)} A(h,q) = A^f(y^1,q)$$

Hence A^f is anti Q- fuzzy R- sub modules of R.

Conclusion: AbuOsman[1] introduced the concept of Direct product of fuzzy subgroups.[6] investigated the concept of lattice valued Q- fuzzy sub modules of near rings. In this paper we investigate the concept of Anti Q-fuzzy right R- sub modules over commutative rings and derive some simple consequences.

Applications: This work has enormous applicability in the diverse disciplines of biosciences (such as bisocosmology, bisoseismology, bis-behavioural sciences and neural networks.

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