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Further Results on the Mediator Chromatic Number

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ABSTRACT
In this paper, we investigate the effect on mediator chromatic number of a graph when certain unary operations are carried out. Further, we discuss the mediator colouring of join of two graphs.

General Terms
Graph colouring, chromatic number,operations on graphs.

Keywords
Mediator colourclass, mediator colouring,mediator chromatic number.

1. INTRODUCTION
Graph colouring belongs to the classical graph theoretical problems that are important both for their practical applications and richness of theoretical results. A k-vertex colouring of a graph G is an assignment of k colours to the vertices of G and it is proper if no two distinct adjacent vertices have the same colour. This fundamental definition together with some special conditions will form sevaral type of colourngs. The mediator chromatic number [7] was introduced by N.Roopesh and K. Thilagavathi. They [8] further proved its influence on a minimum harmonious colouring of a graph.

2. DEFINITIONS [7, 8]

2.1 Mediator Colouring Partition
Let $G(V,E)$ be a simple graph and $\Pi = \{C_1, C_2, C_3, \ldots, C_k\}$ be a proper colouring partition of $V(G)$. This partition is said to be a mediator colouring partition if the following condition is satisfied.

(i) $C_i \cup C_j,\ i \neq j$ is not independent

or

(ii) There exist at least two edges from $C_i \cup C_j$ to some other colour class $C_l, l \neq i, j$

Note that the second choice ensures at least two edges incident to the vertex which is coloured as c_l and the other ends with colours c_i and c_j.

2.2 Mediator colouring
Mediator colouring is a proper vertex colouring in which the colouring partition is a mediator colouring partition.

2.3 Mediator chromatic number
The maximum cardinality of a mediator colouring partition of a graph G is called the mediator chromatic number of the graph. And it is denoted by $\chi_M(G)$.

Note : It can be noted that if the first condition of mediator colouring alone is satisfied for every i and j , then it is called the complete colouring or achromatic colouring and in this case χ_M will become ψ.

Note: We denote the colour classes of the vertices with colour c_i as C_i. Then two colour classes C_i and C_j are adjacent if there exist an edge with end points coloured as c_i and c_j .

2.4 Mediator set
In a mediator colouring, any set C_l, such that there exists at least two edges from $C_i \cup C_j$ to C_l (where $l \neq i, j$) is called a **mediator colour class** of C_i, C_j, and c_l, is called a **mediator** of c_i, c_j. The collection of all mediator colour classes is called the **mediator set** of the mediator colouring.

3. MEDIATOR CHROMATIC NUMBER OF A GRAPH WHEN A VERTEX IS REMOVED

3.1 Theorem
For any graph G and u in V (G), $\chi_M(G) \geq \chi_M(G-u) \geq \chi_M(G) - 1$.

Proof

Let the vertex u be coloured as c. If $\chi_M(G-u) = n$, then G has a mediator n-colouring unless any of the following cases holds.

Case i : c is adjacent with all the other n colours.

Case ii : $\{c\} \cup C_i$, $i = 1,2,\ldots,m$ is not independent, where $\{C_1, C_2, \ldots, C_m\}$ is the mediator set of the mediator colouring.

Case iii : c is adjacent to some of the colours and the mediators of the rest of the colours.

In the above three cases G has a $n+1$ mediator colouring. Thus $\chi_M(G) \geq \chi_M(G-u)$.

On the other hand, suppose $\chi_M(G) = n$, consider the colouring of $G-u$ induced by a mediator n colouring of G,in which u is assigned the colour c. Then any one of the following cases occur.

Case 1: The colouring of $G-u$ is a mediator colouring.

Case 2: The colouring is not a mediator colouring.

If case 1 occur, then $\chi_M(G-u) = n$. Thus, we get $\chi_M(G-u) > \chi_M(G) - 1$ directly. Now if case 2 occur, consider the colouring map □$: V \to \Pi$, where Π is the collection of colour classes. Let □$(u) = c$. Then the induced colouring of $G-u$ is not a mediator implies, $\{c\}$ is a mediator colour class for some $C_i \cup C_j$, where $C_i, C_j \in \Pi$. Now select all the c_i or c_j, which is not adjacent to the colour c and recolour it as c. The result is a $n-1$ mediator colouring of $G-u$.

That is, $\chi_M(G-u) = \chi_M(G) - 1$.

Thus, we get $\chi_M(G-u) \geq \chi_M(G) - 1$.

3.2 Corollary
If $\chi_M(G-u) = \chi_M(G)$, then there is a mediator -n colouring of G which induces a mediator -n colouring of $G-u$.

Proof

Let $\chi_M(G-u)=\chi_M(G)$. Suppose no mediator-n colouring of G induces a mediator -n colouring of $G-u$. Then, in every mediator- n colouring of $G-u$, we find some $C_i \cup C_j$ as independent and also no mediator colour class for this union. Thus, we get $\chi_M(G) \geq 1+\chi_M(G-u)$. That is $\chi_M(G)-1 \geq \chi_M(G-u)$, which is a contradiction to the above theorem. Hence the corollary.

3.3 Corollary

If H is an induced sub graph of G, then $\chi_M(G) \geq \chi_M(H)$.

4. MEDIATOR CHROMATIC NUMBER OF A GRAPH WHEN AN EDGE IS REMOVED.

4.1 Theorem

For a cycle C_n of length greater than 5 and an edge, $e \in V(C_n)$, $\chi_M(C_n-e)=\chi_M(C_n)$.

Proof

Consider a cycle C_n, $n>5$. Since a cycle is a two connected graph, by Whitney's theorem (A graph G having at least 3 vertices is two connected iff for each pair $u,v \in V(G)$ there exist internally disjoint uv- paths in G.), C_n-e is a path of length $n-1$.

Therefore $\chi_M(C_n-e)=\chi_M(P_{n-1})$

By theorem 3.4 in [7], we have for $n>5$, $\chi_M(C_n)=\chi_M(P_{n-1})$. Thus $\chi_M(C_n-e)=\chi_M(P_{n-1})=\chi_M(C_n)$.

After going through the above theorem, the question arises in one's mind as: Is it true for any two connected graph? The restriction, $n>5$ in the above theorem will answer it clearly. Then, what about a two connected graph of order greater than 5? Unfortunately the answer is NO. For, consider the following graph G with $\chi_M(G)=6$. It is obvious that if we remove the edge e its mediator chromatic number reduces by one.

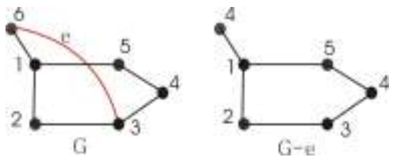

Figure 1: $\chi_M(G-e)=\chi_M(G)-1$

Now for a general graph G and an edge e of G, the mediator chromatic number of the graph $G-e$ can be less than, equal to or greater than the mediator chromatic number of the graph G. From the previous theorem, it is clear that there are graphs for which the equal to condition holds. Now for the rest of the conditions we will give some examples.

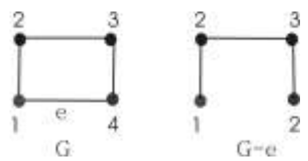

Figure 2: $\chi_M(G-e)<\chi_M(G)$

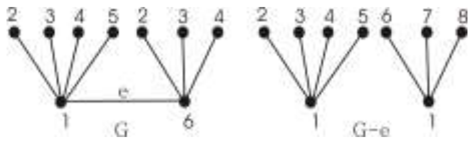

Figure 3: $\chi_M(G-e)>\chi_M(G)$

5. MEDIATOR COLOURING OF JOIN OF TWO GRAPHS

The *join* [9] of two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ is defined as the graph G_1+G_2 consist of $G_1 \cup G_2$ and all lines joining V_1 and V_2.

5.1 Theorem

For any two graphs G_1 and G_2 with order n_1 and n_2, the mediator chromatic number of their join, $\chi_M(G_1+G_2)=n_1+n_2$.

Proof

Let $V(G_1)=\{v_1,v_2,v_3,\dots,v_{n_1}\}$ and $V(G_2)=\{u_1,u_2,u_3,\dots,u_{n_2}\}$. Then $|V(G_1+G_2)|=n_1+n_2$. Assign a colouring to the vertices of G_1+G_2 as follows: Assign the colour c_i to $v_i, i=1,2,3,\dots,n_1$ and c_{n_1+j} to u_j for $j=1,2,3,\dots,n_2$. Now we have to show that this colouring is a mediator colouring. For this,consider the colour class $C=\{c_i\}=C_1 \cup C_2=\{c_j\}\cup\{c_k\}$, $j=1,2,3,\dots,n_1$ and $k=1,2,3,\dots,n_2$ where $\{c_j\}$is the colour class used for colouring G_1 and $\{c_k\}$ is the colour class used for colouring G_2. Now consider any two colour classes C_i and C_j.

Case 1

Let C_i , $C_j \in C_1$. Now if there is an edge in G_1 which connects the vertices which has been coloured as c_i and c_j we are through. Suppose not, then in G_1+G_2, we can find edges from C_i and C_j to each vertices of G_2. Thus, any colour class in the family C_2 will serve as our C_l.

Case 2

Let C_i , $C_j \in C_2$. Again by using the similar arguement in case 1 we can see that there exists edges from C_i and C_J to the vertices of G_1. Thus any colour class in the family C_1 will serve as our C_l.

Case 3

Let $C_i \in C_1$ and $C_j \in C_2$. Then, by the definition of join there exists an edge between the vertices which has been coloured as c_i and c_j. That is $C_i \cup C_j$ is not independent. Thus, in all the cases, we get the given colouring is a mediator colouring and by the very construction, it is the maximum one.

Therefore, $\chi_M(G_1+G_2)=n_1+n_2$.

Example

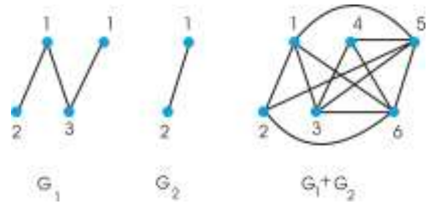

Figure 4: mediator colouringof join of two graphs

5.2 Note

From the above example, it can be noted that, in general, for a graph G, $\chi_M(G_1+G_2) \neq \chi_M(G_1)+\chi_M(G_2)$.

5.3 Corollary

$\chi_M(K_{m,n}) = m+n$

Proof

We can write $K_{m,n} = \overline{K}_m + \overline{K}_n$.

Therefore $\chi_M(K_{m,n}) = \chi_M(\overline{K_m}+\overline{K_n}) = |V(\overline{K_m})| + |V(\overline{K_n})| = m+n$

Thus $\chi_M(K_{m,n}) = m+n$.

5.4 Corollary

For any graph G and the complete graph K_p, $\chi_M(G+K_p) = n+p$

5.5 Corollary

For the wheel graph W_n, $\chi_M(W_n) = n+1$.

Proof

Proof is immediate from the fact $W_n = C_n + K_1$.

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