The Mediator Chromatic Number of Grid Graphs

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ABSTRACT
In this paper, we determine the mediator chromatic number of cartesian product of a path with K_2 and thus prove that the mediator chromatic number of Ladder graphs and Grid graphs are polynomially solvable.

General Terms
Graph colouring, chromatic number, operations on graphs.

Keywords
Mediator colourclass, mediator colouring, mediator chromatic number.

1. INTRODUCTION
In graph theory, graph colouring is an assignment of colours to certain objects in a graph. Such objects can be vertices, edges, faces or a mixture of those. Among these, vertex colouring is the important kind. It is a usual custom to use the numbers $1,2,3,\ldots,k$ to colour a graph instead of the actual colours. A k-vertex colouring of a graph G is an assignment of k colours to the vertices of G and it is proper if no two distinct adjacent vertices have the same colour. In this entire paper by colouring, we mean the proper vertex colouring. The mediator chromatic number [7] was introduced by N.Roopesh and K. Thilagavathi. They [8] shown that it can be used as a tool for checking the minimum of a harmonious colouring.

2. DEFINITIONS [7, 8]
2.1 Mediator Colouring Partition
Let $G(V,E)$ be a simple graph and $\Pi = \{C_1, C_2, C_3, \ldots, C_k\}$ be a proper colouring partition of $V(G)$. This partition is said to be a mediator colouring partition if the following condition is satisfied.

(i) $C_i \cup C_j,\ i \neq j$ is not independent

or

(ii) There exist at least two edges from $C_i \cup C_j$ to some other colour class $C_l, l \neq i,j$

Note that the second choice ensures at least two edges incident to the vertex which is coloured as c_l and the other ends with colours c_i and c_j.

2.2 Mediator colouring
Mediator colouring is a proper vertex colouring in which the colouring partition is a mediator colouring partition.

2.3 Mediator chromatic number
The maximum cardinality of a mediator colouring partition of a graph G is called the mediator chromatic number of the graph. And it is denoted by $\chi_M(G)$.

Note : It can be noted that if the first condition of mediator colouring alone is satisfied for every i and j, then it is called the complete colouring or achromatic colouring and in this case χ_M will become ψ.

Note: We denote the colour classes of the vertices with colour c_i as C_i. Then two colour classes C_i and C_j are adjacent if there exist an edge with end points coloured as c_i and c_j.

2.4 Mediator set
In a mediator colouring, any set C_l, such that there exists at least two edges from $C_i \cup C_j$ to C_l (where $l \neq i,j$) is called a **mediator colour class** of C_i, C_j and c_l is called a **mediator** of c_i, c_j. The collection of all mediator colour classes is called the **mediator set** of the mediator colouring.

3. MEDIATOR COLOURING OF CARTESIAN PRODUCT OF P_n WITH K_2
In graph theory, the Cartesian product [10] $G \times H$ of graphs G and H is a graph defined as follows : vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u,u') and (v,v') are adjacent in $G \times H$ if and only if either $u = v$ and u' is adjacent with v' or $u' = v'$ and u is adjacent with v.

3.1 Theorem
For a path P_n of odd length $n > 1$, $\chi_M(P_n \times K_2) = n + m + 2$ where $n = 2m + 1$.

Proof

Consider a path P_n of odd length n and a complete graph K_2. Let $V(P_n) = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ and $V(K_2) = \{u_1, u_2\}$.Then $V(P_n \times K_2) = \{w_1, w_2, w_3, \ldots, w_{n+1}, x_1, x_2, x_3, \ldots, x_{n+1}\}$, where $w_i = (v_i, u_1)$ and $x_i = (v_i, u_2)$. Consider the colour class $C = \{c_1, c_2, c_3, \ldots, c_{n+m+2}\}$, assign the colours to the vertices of $P_n \times K_2$ as follows

Colour the vertices w_1 as c_1 , x_1 , w_2 , w_{2+4k} and x_{4r} as c_2, where $k, r = 1,2,3,\ldots,[\frac{n}{4}]$. Let $W' = \{w_i\} - \{w_1, w_2, w_{2+4k}\}$. Form a sequence $\{t_i\}$, $t_i \in W'$ such that $\{i\}$ is a strictly increasing sequence. Consider $X' = \{x_i\} - \{x_1, x_{4r}\}$. Form a sequence $\{y_h\}$, $y_h \in X'$ and $\{h\}$ is a strictly increasing sequence. Now for $i = 1,2,3,\ldots,n - [\frac{n}{4}] - 1$ and $h = 1,2,3,\ldots,n - [\frac{n}{4}]$ assign the colour c_{2i+2} to t_i and c_{2h+1} to y_h. Then this colouring is a mediator, for the set of vertices coloured as c_2 will act as the colour class C_l such that for each i,j , $i,j \neq l$ there exists at least two edges from $C_i \cup C_j$ to C_l. More over, it is the maximum colouring. For, suppose not, that is a mediator colouring is possible with a colour class C' with $|C'| = |C| + 1$. Let c be the new colour added to C. Now to get such a colouring, only possibility is to recolour any of the vertex coloured as c_2. That is we have to consider the recolouring of the vertces of the following forms.

(i) $w_{2+4k}, k = 1,2,3,\ldots,[\frac{n}{4}]$

(ii) $x_{4r}, r = 1,2,3,\ldots,[\frac{n}{4}]$

(iii) x_1, w_2

Case 1

Suppose we are recolouring the vertex of the form w_{2+4s}, $1 \leq s \leq [\frac{n}{4}]$, then w_{2+4s} is adjacent with at least two vetices of the graph. Consider any one such vertex say w . Now let $z \in N(w)$, and $a \in N(Z)$, $a \neq w$.

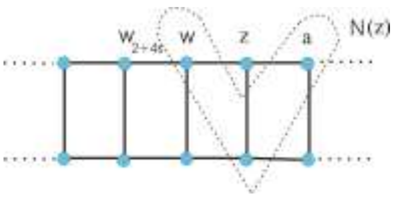

Figure 1: showing the selection of a

Consider the colouring map $\square: V \to C'$ then $\{\square(w_{2+4s})\} \cup \{\square(a)\}$ is independent. Then to make this colouring mediator, we need a different colour class C_l such that there exists at least two edges from $\{\square(w_{2+4s})\} \cup \{\square(a)\}$ to C_l. Since we are going for a maximum colouring, the only way is to recolour the neighbour of a, which is coloured as c_2 to c. Then again using the previous arguement, we can find a vertex b such that $\{\square(w_{2+4s})\} \cup \{\square(b)\}$ is independent. Thus, the recolouring process of c_2 to c will continue upto the last vertex with colour c_2. Now the colour class will become $C' - \{c_2\}$. Thus $|C'| = |C| + 1 - 1 = |C|$, which is a contradiction. Therefore C is the maximum colour class.

Case 2

If we are recolouring the vertex of the form x_{4s}, $1 \leq s \leq [\frac{n}{4}]$,by using a similar arguement used in case 1, we can prove that C is the maximum colour class possible.

Case 3

If we are recolouring x_1 and w_2 , we can easily see that the vertices $w_3, w_4, \ldots, w_{n+1}$, $x_4, x_5, \ldots, x_{n+1}$ are not adjacent with x_1 or w_2. Thus, again we can choose a vertex a such that $\{c\} \cup \{\square(a)\}$ is independent, then we can proceed as in case 1 to prove the colour class C is the maximum one.

It can be noted that any recolouring of the combination of the vertices amoung (i), (ii) and (iii) will also enable us to select a vertex a such that $\{c\} \cup \{\square(a)\}$is independent.

Thus in all the cases, we get the colourclass C is the maximum one possible.

Hence for odd n, $\chi_M(P_n \times K_2) = n + m + 2$ where $n = 2m + 1$.

Example

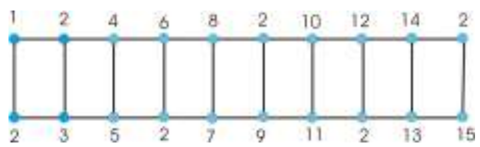

Figure 2: $\chi_M(P_9 \times K_2) = 9 + 4 + 2 = 15$

3.2 Note

For $n = 1$, $\chi_M(P_1 \times K_2) = 4$

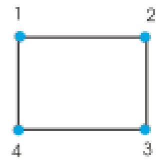

Figure 3: $\chi_M(P_1 \times K_2) = 4$

3.3 Theorem

For a path P_n of even length,$\chi_M(P_n \times K_2) = n + \frac{n}{2} + 2$.

Proof

Consider a path P_n of even length n and a complete graph K_2 . Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n, v_{n+1}\}$ and $V(K_2) = \{u_1, u_2\}$. Then $V(P_n \times K_2) = \{w_1, w_2, w_3, \ldots, w_{n+1}, x_1, x_2, x_3, \ldots, x_{n+1}\}$, where $w_i = (v_i, u_1)$ and $x_i = (v_i, u_2)$. Consider the colour class $C = \{c_1, c_2, c_3, \ldots, c_{n+\frac{n}{2}+2}\}$, and assign the colours to the vertices of $P_n \times K_2$ as follows.

Case 1: n is a multiple of 4.

Colour the vertices w_1 as c_1 and each vertex of the form w_{3+4k} or x_{1+4r} as c_2, where $k = 0,1,2, \ldots, \frac{n}{4} - 1$ and $r = 0,1,2, \ldots, \frac{n}{4}$. Let $W' = \{w_1, w_2, \ldots, w_{n+1}\} - \left(\{w_1\} \cup \{w_{3+4k}: k = 0,1,2, \ldots, \frac{n}{4} - 1\}\right)$. Let $(t_i)_{i=1}^{\frac{3n}{4}}$ be the sequence of elements from W' in order of increasing index.

Let $X' = \{x_1, x_2, \ldots, x_{n+1}\} - \left(\{x_{1+4r}: r = 0,1,2, \ldots, \frac{n}{4}\}\right)$. Let $(y_h)_{h=1}^{\frac{3n}{4}}$ be the sequence of elements from X' in order of increasing index. Now for $i, h = 1,2,3, \ldots, \frac{3n}{4}$ assign the colour c_{2i+2} to t_i and c_{2h+1} to y_h.

Case 2: n is not a multiple of 4.

Colour w_1 as c_1 and each vertex of the form w_{3+4k} or x_{1+4r} as c_2, where $k, r = 0,1,2, \ldots, \left\lceil \frac{n}{4} \right\rceil$. Similar to X' consider the set W',

$W' = \{w_1, w_2, \ldots, w_{n+1}\} - \left(\{w_1\} \cup \left\{w_{3+4k}: k = 0,1,2, \ldots, \left\lceil \frac{n}{4} \right\rceil\right\}\right)$. Let $(t_i)_{i=1}^{\frac{3n}{4}}$ be the sequence of elements from W' in order of increasing index. Let $X' = \{x_1, x_2, \ldots, x_{n+1}\} - \left(\{x_{1+4r}: r = 0,1,2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}\right)$. Let $(y_h)_{h=1}^{\left\lceil \frac{3n}{4} \right\rceil + 1}$ be the sequence of elements from X' in order of increasing index. Now for $i = 1,2,3, \ldots, \left\lceil \frac{3n}{4} \right\rceil$ and $h = 1,2,3, \ldots, \left\lceil \frac{3n}{4} \right\rceil + 1$ assign the colour c_{2i+2} to t_i and c_{2h+1} to y_h.

Then this colouring is a mediator, for the set of vertices coloured as c_2 will act as the colour class C_l such that for each i, j such that $i, j \neq l$ there exists at least two edges from $C_i \cup C_j$ to C_l. That is, $\{C_2\}$ will act as the mediator set. Moreover, it is the maximum colouring. For suppose not, that is, a mediator colouring is possible with a colour class C' with $|C'| = |C| + 1$. Let $C' = C \cup \{c\}$. To get such a colouring, the only possibility is to recolour the c_2 coloured vertices as c. That is, we have to consider the recolouring of some vertices of the following forms.

(i) w_{3+4k}, where $k = \begin{cases} 0,1,2, \ldots, \frac{n}{4} - 1, \text{if n} \equiv 0 (\text{mod } 4) \\ 0,1,2, \ldots, \left\lceil \frac{n}{4} \right\rceil, \text{ otherwise} \end{cases}$

(ii) x_{1+4r}, where $r = \begin{cases} 0,1,2, \ldots, \frac{n}{4}, \text{if n} \equiv 0 (\text{mod } 4) \\ 0,1,2, \ldots, \left\lceil \frac{n}{4} \right\rceil, \text{otherwise} \end{cases}$

Case ($\boldsymbol{i}$)
Suppose we are recolouring some vertex w of the form w_{3+4k}, then it is adjacent to atleast two vertices of the graph. Let $z \in N(w)$, and $a \in N(z)$, $a \neq w$. Consider the colouring map $\square: V \rightarrow C'$ then $C_{\square(w)} \cup C_{\square(a)}$ is independent,where C_d is the colour class corresponding to colour d. Then to make this colouring a mediator, we need a colour class C_l such that there exists atleast two edges from $C_{\square(w)} \cup C_{\square(a)}$ to C_l. Since we are going for a maximum colouring, the only way is to recolour the neighbour of a, which is coloured as c_2 to c. Now starting with either vertex w or w'(the recoloured vertex), and again using the previous argument, we can find another vertex b such that $C_{\square(w)} \cup C_{\square(b)}$ is independent. Thus, the recolouring process from c_2 to c will continue upto the last vertex with colour c_2, which results in expelling the colour c_2 from the colour class C' . Thus, $|C'| = |C| + 1 - 1 = |C|$, which is a contradiction. Therefore, C is the maximum colour class.
Case ($\boldsymbol{ii}$)
If we are recolouring some vertex of the form x_{4r+1}, by a similar arguement as Case (i), we can prove that C is the maximum colour class possible.
It can be noted that any recolouring of the combination of vertices in case (i) and case (ii) will also enable us to select a vertex a such that $C_c \cup C_{\square(a)}$ is independent, or in expelling one already existing colour and thus reducing the cardinality of the new colour class by one.
Thus in all the cases C is the maximum colour class possible.
Hence for even n, $\chi_M(P_n \times K_2) = n + \frac{n}{2} + 2$

4. CONCLUSION
The graph $P_n \times K_2$ is generally known as the *ladder graph* L_{n+1}. Also it is equivalent to the *grid graph* $G_{2,n+1}$. By theorem 3.1, for odd n the mediator chromatic number of L_{n+1} and $G_{2,n+1}$ is $n + m + 2$ where $n = 2m + 1$. For even n, the mediator chromatic number of L_{n+1} and $G_{2,n+1}$ is given by theorem 3.2 as $n + \frac{n}{2} + 2$.
Thus, we conclude that for any $n > 1$, $\chi_M(P_n \times K_2) = n + [\frac{n}{2}] + 2$.
Hence, $\chi_M(L_{n+1})$=$\chi_M(G_{2,n+1}) = n + [\frac{n}{2}] + 2$.

Example

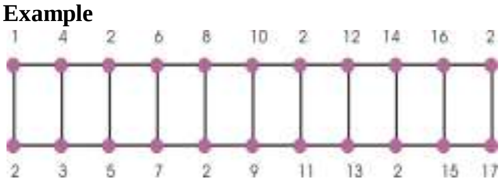

Figure 4: $\boldsymbol{\chi_M(P_{10} \times K_2) = 10 + 5 + 2 = 17}$

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