### Non Linear Dynamics of Ishikawa Iteration

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#### ABSTRACT

We introduce in this paper the dynamics for Ishikawa iteration procedure. The geometry of Relative Superior Mandelbrot sets are explored for Ishikawa iterates.

#### **Keywords**

Complex dynamics, Relative Superior Mandelbrot Set, Ishikawa Iteration..

#### **1. INTRODUCTION**

Complex graphics of nonlinear dynamical systems have been a subject of intense research nowadays. These graphics are generally obtained by "coloring" the escape speed of the seed points within the certain regions of the complex plane that give rise to the unbounded orbits. The complexity of the mathematical objects such as Julia sets and Mandelbrot sets, in spite of their deceitful simplicity of equations that generate them is truly overwhelming.

Perhaps, the Mandelbrot set is the most popular object in the fractal theory. It is believed to be the most beautiful object not only in the real but also in the complex plane. This object was given by Benoit B. Mandelbrot in 1979 and has been the subject of intense research right from its advent. Mandelbrot set and its various extensions and variants have been extensively studied using Picard's iterations.

Recently M. Rani and V. Kumar[21] introduced the superior Mandelbrot sets using Mann iteration procedure. We introduce in this paper a new class of Mandelbrot sets named as Relative Superior Mandelbrot sets using Ishikawa iterations. Our study shows that Relative Superior Mandelbrot sets are exclusively elite and effectively different from other Mandelbrot sets existing in the present literature.

#### 2. PRELIMINARIES

Let  $\{z_n : n = 1, 2, 3, 4, \dots\}$ , denoted by  $\{z_n\}$  be a sequence of complex numbers. Then, we say  $\lim_{n \to \infty} z_n = \infty$  if for given M > 0, there exists N > 0, such that for all n > N, we must have  $|z_n| > M$ . Thus all the values of  $z_n$ , lies outside a circle of radius M, for sufficiently large values of n.

Let  $Q(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z^1 + a_n z^0; a_0 \neq 0$ 

be a polynomial of degree n, where  $n \ge 2$ . The coefficients are allowed to be complex numbers. In other words, it follows that  $Q_c(z) = z^2 + c$ .

**Definition 2.1:** Let X be a nonempty set and  $f:X\to X$  . For any point  $x_0\in X$  , the Picard's orbit is defined as the set of

iterates of a point  $X_0$ , that is;

$$O(f, x_0) = \{x_n; x_n = f(x_{n-1}), n = 1, 2, 3....\}.$$

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number. The collection of points that are bounded, *i.e.* there exists M, such that  $|Q^n(z)| \leq M$ , for all n, is called as a prisoner set while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Mandelbrot set for Q.

**Definition 2.2:** The Mandelbrot set M for the quadratic  $Q_c(z) = z^2 + c$  is defined as the collection of all  $c \in C$  for which the orbit of the point 0 is bounded, that is

$$M = \{c \in C : \{Q_c^n(0)\}; n = 0, 1, 2, \dots \text{ is bounded}\}$$

An equivalent formulation is

 $M = \{c \in C : \{Q_c^n(0) \text{ does not tend to } \infty \text{ as } n \to \infty\}$ 

We choose the initial point 0, as 0 is the only critical point of  ${\cal Q}_c$  .

### 3. ISHIKAWA ITERATION FOR RELATIVE SUPERIOR MANDELBROT SETS

Let X be a subset of real or complex numbers and  $f: X \to X$ . For  $x_0 \in X$ , we construct the sequences  $\{x_n\}$  and  $\{y_n\}$  in X in the following manner:

$$y_0 = s'_0 f(x_0) + (1 - s'_0) x_0$$
  

$$y_1 = s'_1 f(x_1) + (1 - s'_1) x_1 \dots$$
  

$$y_n = s'_n f(x_n) + (1 - s'_n) x_n$$

where  $0 \le s'_n \le 1$  and  $s'_n$  is convergent to non zero number and

$$x_{1} = s_{0} f(y_{0}) + (1 - s_{0}) x_{0}$$
  

$$x_{2} = s_{1} f(y_{1}) + (1 - s_{1}) x_{1} \dots$$
  

$$x_{n} = s_{n-1} f(y_{n-1}) + (1 - s_{n-1}) x_{n-1}$$

where  $0 \le s_n \le 1$  and  $s_n$  is convergent to non zero number[12].

**Definition 3.1:** The sequences  $x_n$  and  $y_n$  constructed above is called Ishikawa sequences of iterations or relative superior sequences of iterates. We denote it by  $RSO(x_0, s_n, s'_n, t)$ .

Notice that  $RSO(x_0, s_n, s_n', t)$  with  $s_n'=1$  is  $RSO(x_0, s_n, t)$ 

i.e. Mann's orbit and if we place  $s_n = s'_n = 1$  then

 $RSO(x_0, s_n, s'_n, t)$  reduces to  $O(x_0, t)$ .

We remark that Ishikawa orbit  $RSO(x_0, s_n, s'_n, t)$  with  $s'_n = 1/2$  is Relative Superior orbit.

Now we define Mandelbrot sets for function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets

**Definition 3.2:** Relative Superior Mandelbrot set RSM for the function of the form  $Q_c(z) = z^n + c$ , where n = 1, 2, 3, 4... is

defined as the collection of  $c \in C$  for which the orbit of 0 is bounded *i.e.* 

 $RSM = \{c \in C : Q_c^k(0) : k = 0, 1, 2...\}$  is bounded.

We now define escape criterions for these sets.

#### 3.1 Relative Superior Escape Criterions for Quadratics:

The following theorem gives us an escape Criterions for function  $Q_c = z^2 + c$  in respect to Ishikawa iteration procedure.

**Theorem 3.1:** Let's assume that  $|z| \ge |c| > 2/s$ ;  $|z| \ge |c| > 2/s'$ , where 0 < s < 1, 0 < s' < 1 and c is a complex number.

Define 
$$z_1 = (1-s)z + sQ_c(z)$$
  
:  
 $z_n = (1-s)z_{n-1} + sQ_c(z_{n-1})$ 

Where  $Q_c(z)$  can be a quadratic, cubic or biquadratic polynomial in terms of s' and n = 2,3,4,...,

)

then 
$$|z_n| \to \infty$$
, as  $n \to \infty$ .

**Proof:** Let's take  $|Q_c(z)| = |(1-s')z + s'Q'_c(z)|$ ,

where 
$$Q'_c(z) = z^2 + c$$

$$= |s'z^{2} + (1-s')z + s'c|$$

$$\geq |s'z^{2} + (1-s')z| - |s'c|$$

$$\geq |z| (|s'z + (1-s')|) - s'|z| \quad (\because |z| \ge |c|)$$

$$\geq |z| (|s'z| - 1 + s') - s'|z|$$

$$= |z| (|s'z| - 1) \dots (1)$$
Now since,  $z_{n} = (1-s)z_{n-1} + sQ'_{c}(z)$ 
So,  $|z_{1}| = |(1-s)z + sQ_{c}(z)|$  on substituting (1)  

$$= |(1-s)z + s|z| (|s'z| - 1)|$$

$$= |z - sz + s|z| . |s'z| - s|z||$$

$$\geq (|z| + |sz|) + (s|z| . |s'z| - s|z|)$$

$$\geq |z| + |sz| + s|z| . |s'z|$$

$$\geq |z| + |s|z| . |s'z|$$

$$\geq |z| (1 + ss'|z|), \text{ since } s|z| > 2,$$

so,  $ss' \mid z \mid \geq 2$ , there exists  $\lambda \geq 0$ , such that  $ss' \mid z \mid -1 > 1 + \lambda$ 

Consequently  $|z_1| > (1+\lambda) |z|$ 

$$|z_n| > (1+\lambda)^n |z|$$

Thus, the Ishikawa orbit of z, under the quadratic function tends to infinity. This completes the proof.

**Corollary 3.1:** Suppose that |c| > 2/s; |c| > 2/s'. Then, the Relative Superior orbit of Ishikawa  $RSO(Q_c, 0, s, s')$  escapes to infinity.

In the proof of the theorem, we used the facts that  $|z| \ge |c|$  and |z| > 2/s as well as |z| > 2/s'. Hence, the following corollary is the refinement of the escape criterion discussed in the above theorem.

**Corollary 3.2(Escape Criterion):** Suppose that  $|z| > \max\{|c|, 2/s, 2/s'\}$ , then  $|z_n| > (1+\lambda)^n |z|$  and  $|z_n| \to \infty$  as  $n \to \infty$ .

**Corollary 3.3:** Suppose that  $|z_k| > \max\{|c|, 2/s, 2/s'\}$ , for some  $k \ge 0$ . Then,  $|z_{k+1}| > (1+\lambda)^n |z_k|$  and  $|z_n| \to \infty$  as  $n \to \infty$ .

This corollary gives us an algorithm for computing the Relative Superior Mandelbrot sets of  $Q_c$ , for any c. Given any point  $|z| \leq |c|$ , we have computed the Relative superior orbit of z. If for some n,  $|z_n|$  lies outside the circle of radius  $\max\{|c|, 2/s, 2/s'\}$ , we guarantee that the orbit escapes. Hence, z is not in the Relative Superior Mandelbrot sets. On the other hand, if  $|z_n|$  never exceeds this bound, then by definition of the Relative Superior Mandelbrot sets, denoted by RSM. We can make extensive use of this algorithm in the next section.

## 3.2 Relative Superior Escape Criterion for Cubic Polynomials:

We prove the following theorem for the function  $Q_{a,b}(z) = z^3 + az + b$  with respect to the Ishikawa iteration

**Theorem 3.2:** Suppose  $|z| > |b| > (|a|+2/s)^{1/2}$ ,  $|z| > |b| > (|a|+2/s')^{1/2}$  exists, where  $0 < s \le 1; 0 < s' < 1$ and *a* and *b* are in complex plane .Define  $z_1 = (1-s)z + sQ_{a,b}(z)$ .

 $z_n = (1-s)z_{n-1} + sQ_{a,b}(z_{n-1})$ , n =2, 3.....

where  $Q_{a,b}(z)$  is the function of s', then  $|z_n| \to \infty$  as  $n \to \infty$ .

**Proof:** Let's take  $|Q_{a,b}(z)| = |(1-s')z + s'Q'_{a,b}(z)|$ 

$$= |(1-s')z + s'(z^{3} + az + b)|$$

$$= |s'z^{3} + s'az + z - s'z + bs'|$$

$$\geq |s'z^{3} + s'az + z - s'z| - |bs'|$$

$$\geq |z|(|s'z^{2} + s'a + 1 - s'|) - s'|z| \qquad \because |z| \geq |b|$$

$$\geq |z|(|s'z^{2} + as'| - |1 - s'|) - s'|z|$$

$$= |z|\{|s'z^{2} + as'| - 1 + s' - s'\}$$

$$= |z|\{s'|z^{2} + a| - 1\}$$

$$= s'|z|\{|z^{2} + a| - 1/s'\}$$

$$\geq s'|z|\{|z|^{2} - |a| - 1/s'\}$$

$$= s'|z|\{|z|^{2} - (|a| + 1/s')\}$$
Now since,  $|z_{1}| = |(1 - s)z + sQ_{a,b}(z)|$ 

$$= |(1-s)z+s.|z|.\{s'(|z^{2}+a|)-1\}|$$

$$= |z-sz+s|z|.s'(|z^{2}+a|)-s|z||$$

$$\ge |z|+s|z|+\{s|z|.s'(|z^{2}+a|)-s|z|$$

$$\ge |z|+s|z|+s.s'|z|(|z^{2}+a|)-s|z|$$

$$\ge |z|\{1+s.s'|z^{2}+a|\}$$

$$= |z|s.s'(1/s.s'+|z^{2}+a|)$$

$$\ge |z|s.s'(1/s.s'+|z^{2}|-|a|)$$

$$= |z|s.s'\{|z^{2}|-(|a|-1/s.s')\}$$
Since  $|z| > (|a|+2/s)^{1/2}$  and  $|z| > (|a|+2/s')^{1/2}$  exists and so  $|z| > (|a|+2/s')^{1/2}$  follows .Therefore,  $|z^{2}|-(|a|+1/ss')^{1/2} > 1/ss'$  such that  $ss'\{|z^{2}|-(|a|+1/ss')\} > 1$ . Hence, there exists  $\gamma > 1$ , such that  $|z_{1}| > \gamma |z|$ . Repeating this argument n time, we get  $|z_{n}| > \gamma^{n} |z|$ . Therefore, the Relative superior orbit of z, under the cubic polynomial  $Q_{a,b}(z)$ , tends to infinity. This completes the proof.

**Corollary 3.4:** Suppose that  $|b| > (|a|+2/s)^{1/2}$  and  $|b| > (|a|+2/s')^{1/2}$  exists. Then, the Relative superior orbit  $RSO(Q_{a,b}, 0, s, s')$  escapes to infinity.

#### Corollary 3.5(Escape Criterion): Suppose

 $|z| > \max\{|b|, (|a|+2/s)^{1/2}, (|a|+2/s')^{1/2}\}$ then  $|z_n| \to \infty$  as  $n \to \infty$ . Corollary 3.5 gives an escape criterion for cubic polynomials.

#### Corollary 3.6: Assume that

 $|z_k| > \max\{|b|, (|a|+2/s)^{1/2}, (|a|+2/s')^{1/2}\}$  for some  $k \ge 0$ . Then  $|z_{k+1}| > \gamma |z_k|$  and  $|z_n| \to \infty$ , as  $n \to \infty$ . From Corollary 3.6, we find an algorithm for computing the Relative Superior Mandelbrot sets of  $Q_{a,b}(z)$ , for any a and b.

#### 3.3 A General Escape Criterion:

We will obtain a general escape criterion for polynomials of the form  $G_c(z) = z^n + c$  .

**Theorem 3.3:** For general function  $G_c(z) = z^n + c$ , n = 1, 2, 3, 4 ...where  $0 < s \le 1, 0 < s' < 1$  and c is the complex plane. Define  $z_1 = (1-s)z + sG_c(z)$  $\vdots$  $z_n = (1-s)z_{n-1} + sG_c(z_{n-1})$ 

Thus. the general escape criterion is  $\max\{|c|, (2/s)^{\frac{1}{n+1}}, (2/s')^{\frac{1}{n+1}}\}$ **Proof:** We shall prove this theorem by induction: For n = 1, we get  $G_c(z) = z + c$ . So, the escape criterion is |c|, which is obvious, *i.e.*  $|z| > \max\{|c|, 0, 0\}$ For n = 2, we get  $G_c(z) = z^2 + c$ . So, the escape criterion is  $|z| > \max\{|c|, 2/s, 2/s'\}$  (Theorem 3.1) For n = 3, we get  $G_c(z) = z^3 + c$ . So, the result follows from Theorem 3.2 with a = 0 and b = c, such that the escape criterion is  $|z| > \max\{|c|, (2/s)^{1/2}, (2/s')^{1/2}\}.$ Hence. the theorem is true for n = 1, 2, 3, 4Now, suppose that theorem is true for any n. Let  $G_c(z) = z^{n+1} + c$  and  $|z| \ge |c| > (2/s)^{1/n+1}$  as well as  $|z| \ge |c| > (2/s')^{1/n+1}$  exists. Then,  $|G_n(z)| = |(1-s')z + s'G'_c(z)|$ , where  $G'_c(z) = z^{n+1} + c$  $= |z - s'z + s'(z^{n+1} + c)|$  $\geq |s'z^{n+1} - s'z + z| - s'|c|$  $\geq |z|(s'|z^n|-s'+1|-s'|z|)$  $( :: |z| \ge |c|)$  $\geq |z| \{ (s' | z^n | + | s' | - |1| \} - s' |z| \}$  $\geq |z|(s'|z^n|+s'-1-s')$  $\geq |z|(s'|z^n|-1)$ Now,  $|z_1| = |(1-s)z + sG_n(z)|$  $= |(1-s)z+s|z|(s'|z^n|-1)|$  $= |z - sz + s| |z| . s' |z^{n}| - s |z||$  $\geq (|z|-s|z|)+(ss'|z^{n+1}|-s|z|)$  $\geq (ss' | z^{n+1} | + | z |)$  $\geq |ss'z^{n+1}| - |z|$  $\geq |z|(ss'|z^n|-1)$ Since  $|z| > (2/s)^{1/n}$ ;  $|z| > (2/s')^{1/n}$  and  $|z| > (2/ss')^{1/n}$ .So  $|z| > (2/ss')^{1/n}$ , therefore  $(ss' |z^n| - 1) > 1$ Hence, for some  $\lambda > 0$ , we have  $(ss' \mid z^n \mid -1) > 1 + \lambda$ . Thus,  $|z_1| > (1+\lambda) |z|$  $|z_n| = (1+\lambda)^n |z|$ 

Therefore, the Ishikawa orbit of z under the iteration of  $z^{n+1} + c$  tends to infinity. Hence  $|z| > \max\{|c|, (2/s)^{1/n}, (2/s')^{1/n}\}$  is the escape criterion. This proves the theorem. **Corollary 3.7:** Suppose that  $|c| > (2/s)^{1/n-1}$  and  $|c| > (2/s')^{1/n-1}$  exists. Then, the Relative Superior orbit  $RSO(G_c, 0, s, s')$  escapes to infinity.

Assume

#### Corollary3.8:

that  $|z_k| > \max\{|c|, (2/s)^{1/k-1}, (2/s')^{1/k-1}\}$  for some  $k \ge 0$ . Then  $|z_{k+1}| > \gamma |z_k|$ 

and  $|z_n| \to \infty$ , as  $n \to \infty$ . This corollary gives an algorithm for computing the Relative Superior Mandelbrot sets for the functions of the form  $G_c(z) = z^n + c$ , n = 1, 2, 3, 4...

# 4. GENERATION OF RELATIVE SUPERIOR MANDELBROT SETS:

We generate Relative Superior Mandelbrot sets. We present here some Relative Superior Mandelbrot sets for quadratic, cubic and biquadratic function.

4.1 Relative Superior Mandelbrot Sets for Quadratic function:

Figure 1: Relative Superior Mandelbrot Set for s=s'=1



Figure 2: Relative Superior Mandelbrot Set for s=1, s'=0.3

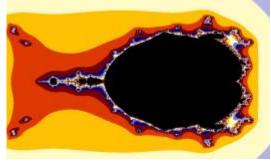


Figure 3: Relative Superior Mandelbrot Set for s=0.3, s'=1

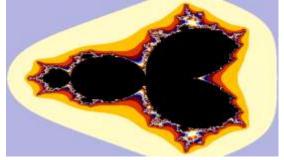
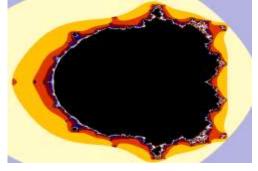


Figure 4: Relative Superior Mandelbrot Set for s=0.1,s'=0.4

Figure 5: Relative Superior Mandelbrot Set for s=0.4, s'=0.1



4.2 Relative Superior Mandelbrot Sets for Cubic function:

Figure 1: Relative Superior Mandelbrot Set for s=s'=1

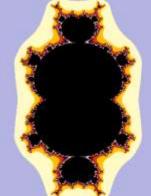
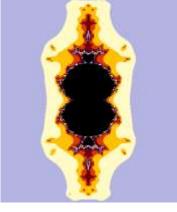


Figure 2: Relative Superior Mandelbrot Set for s=1, s'=0.5



#### Figure 3: Relative Superior Mandelbrot Set for s=0.3 s'=1

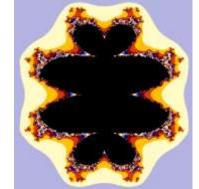


Figure 4: Relative Superior Mandelbrot Set s=0.1, s'=0.4

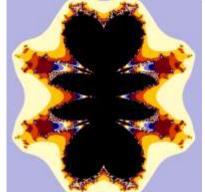
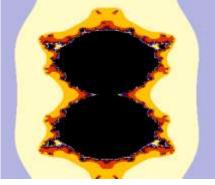


Figure 5: Relative Superior Mandelbrot Set for s=0.4, s'=0.1



**4.3 Relative Superior Mandelbrot Sets for Bi-quadratic function:** 

Figure 1: Relative Superior Mandelbrot Set for s=s'=1

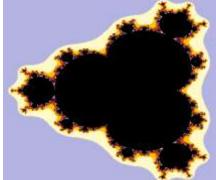


Figure 3: Relative Superior Mandelbrot Set for s=0.5, s'=1

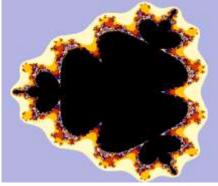


Figure 4: Relative Superior Mandelbrot Set for s=0.1, s'=0.4

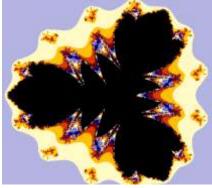
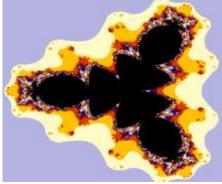


Figure 5: Relative Superior Mandelbrot Set for s=0.3, s'=0.4



4.4 Generalization of Relative Superior Mandelbrot Set

#### Figure 1: Relative Superior Mandelbrot Set for s=s'=1,n=19

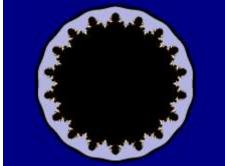


Figure 2: Relative Superior Mandelbrot Set for s=0.1, s'=0.4,

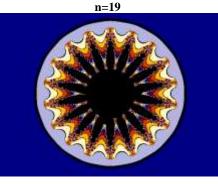


Figure 3: Relative Superior Mandelbrot Set for s=0.4, s'=0.1,

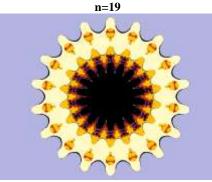


Figure 4: Relative Superior Mandelbrot Set for s=0.1, s'=0.4,

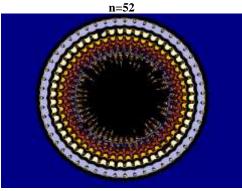
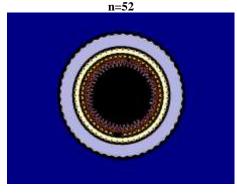


Figure 2: Relative Superior Mandelbrot Set for s=1, s'=0.3

Figure 5: Relative Superior Mandelbrot Set for s=0.4, s'=0.1,

n=52

Figure 6: Relative Superior Mandelbrot Set for s=0.3, s'=0.5,



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