Mandel-Bar Sets of Inverse Complex Function

Yashwant S Chauhan
Asst. Professor
Computer Science & Engg. Department
G. B. Pant Engg. College,
Pauri Garhwal (India)

Rajeshri Rana
Asst. Professor
Applied Science and Humanities Department
G. B. Pant Engg. College,
Pauri Garhwal (India)

Ashish Negi
Asst. Professor
Computer Science & Engg. Department
G. B. Pant Engg. College,
Pauri Garhwal (India)

ABSTRACT

We introduce in this paper the dynamics of Relative Superior Mandel-bar sets of inverse complex function for Ishikawa iteration. The *z* plane fractal images generated from the generalized transformation function $z \to (z^n + c)^{-1}$ for $n \geq 2$ are analyzed.

Keywords: Complex dynamics, Ishikawa Iteration, Relative Superior Mandel-bar sets

1. INTRODUCTION

The object Mandelbrot set given by Mandelbrot in 1979 and its relative object Julia set have become a wide and elite area of research nowadays due to their beauty and complexity of their nature.

Several papers have used escape-time methods to produce images of fractals based on the complex mapping $z \to (z^n + c)^{-1}$, where exponent *n* is a positive integer. Recently, the generalized transformation function $z \to z^{-n} + c$ for positive integer values of n has been considered by K. W. Shirriff [20].On the other hand, Shizuo [14], has presented the various properties of Multicorns and Tricorns for simple complex function, $z \to z^2 + c$ where z and c are complex quantities. Shizuo[15] has also quoted the Multicorns as the generalized Tricorn or the Tricorn of higher order.

The dynamics of antipolynomial $z \to \bar{z}^d + c$ of complex polynomial $z^d + c$, where $d \geq 2$, leads to interesting Tricorn and Multicorns antifractals with respect to function iteration (see [6] and [14, 15]). Multicorns are symmetrical objects. Their symmetry has been studied by Lau and Schleicher [11].

The study of connectedness locus for antiholomorphic polynomials $\bar{z}^2 + c$ defined as Tricorn, coined by Milnor, plays intermediate role between quadratic and cubic polynomials. Crowe etal.[4] considered it as in formal analogy with Mandelbrot set and named it as Mandel-bar set and also brought its features bifurcations along axes rather than at points. Milnor [13] found it as a real slice of cubic connected locus. Winters [21] showed it as boundary along the smooth arc.

In this paper, we investigate the dynamics of the Mandel-bar set for the transformation of the function $z \to (z^n + c)^{-1}$, for $n \geq 2$, and analyze the z plane fractal images generated from the iterations of this function using Ishikawa iteration procedure and analyze the drastic changes that occur in the visual characteristics of the images from *n* = 2, 3, 4,...

2. PRELIMINEARIES

2.1 Mandel-bar Set:

Following the Milnor's [13] study, Shizuo [14] has defined the Tricorn, as the connectedness locus for antiholomorphic polynomials, $z'^n + c$, where $n = 2$. We here define the Mandel-bar set as follows:

Definition2.1: The Mandel-bar set A_c , for the quadratic $A_c(z) = z'^n + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is, $A_c = \{c \in C: A_c\,(0)_{n=0,1,2,3,...}\; is\,bounded\}$.

An equivalent formulation is $A_c = \{c \in C: A_c\,(0)\;not\,tends\,to\;\infty\;as\;n \to \infty\}$

As quoted by Devaney [6], iterations of the function $A_c = z'^2 + c$, using the Escape Time Algorithm, results in many strange and surprising structures. Devaney [6] has named it Tricorn and observed that $f(z')$, the conjugate function of $f(z)$, is antipolynomial. Further, its second iterates is a polynomial of degree 4. Taking the initial choice z_0, one can iterate $A_c^1(z)$, resulting z_1 equals $z_0'^2 + c$, which can be written as $\{|z_0|^2 / z_0\}^2 + c$, since $z_0' * z_0$ is equivalent to $\{|z_0|^2\}$, which gives z_1 equals

$\{|z_0|^4/z_0{}^2\}+c$. Using this value one can state the conjugate of z_1 as $z_1'=\{|z_0|^4/z_0'^2\}+c'$, resulting $z_0{}^2+c'$. Now the second iterate can be stated as $A^2{}_c(z)$ which is equal to $z_1'^2+c$. On simplifying, one can get $\{z_0{}^2+c'\}^2+c$. Further $z_0{}^4+2z_0{}^2c'+c'^2+c$, which is a polynomial of degree 4 in *z*. The critical point for A_c is 0, since $c=A_c(0)$ has only one preimage whereas any other $w\in C$, has two preimages.

Definition2.2: Ishikawa Iteration [10]: Let X be a subset of real or complex numbers and $f:X\to X$. For $x_0\in X$, we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_n=s_n'f(x_n)+(1-s_n')x_n$$

$$x_{n+1}=s_nf(y_n)+(1-s_n)x_n$$

where $0\le s_n'\le 1$, $0\le s_n\le 1$ and s_n' & s_n are both convergent to non zero number.

Definition 2.3[19]: The sequences x_n and y_n constructed above is called Ishikawa sequences of iterations or Relative superior sequences of iterates. We denote it by $RSO(x_0,s_n,s_n',t)$.

Notice that $RSO(x_0,s_n,s_n',t)$ with s_n'=1 is $RSO(x_0,s_n,t)$ *i.e.* Mann's orbit and if we place $s_n=s_n'=1$ then $RSO(x_0,s_n,s_n',t)$ reduces to $O(x_0,t)$.

We remark that Ishikawa orbit $RSO(x_0,s_n,s_n',t)$ with $s_n'=1/2$ is Relative Superior orbit. Now we define Mandelbrot sets for function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets.

Definition 2.4[19]: Relative Superior Mandelbar set RSMB for the function of the form $Q_c(z)=z^n+c$, where n = 1, 2, 3, 4… is defined as the collection of $c\in C$ for which the orbit of 0 is bounded *i.e.*

$RSMB=\{c\in C:Q_c^k(0):k=0,1,2...\}$ is bounded.

Here we present the study of Relative Superior Mandelbar set and Relative Superior Julia set by using the Escape Time Algorithm with respect to Ishikawa Iterates.

Now, we define escape criterions for these sets.

2.4 Escape Criterion: Fractals have been generated from $z\to z^{-n}+c$ using escape-time techniques, for example by Gujar etal. [7, 8] and Glynn [9]. We have used in this paper escape time criteria of Relative Superior Ishikawa iterates for function $z\to(z^n+c)^{-1}$.

We obtain here a general escape criterion for polynomials of the form $G_c(z)=z^n+c$

Escape Criterion for Quadratics: Suppose that $|z|>\max\{|c|,2/s,2/s'\}$, then $|z_n|>(1+\lambda)^n|z|$ and $|z_n|\to\infty$ as $n\to\infty$. So, $|z|\ge|c|$ and $|z|>2/s$ as well as $|z|>2/s'$ shows the escape criteria for quadratics.

Escape Criterion for Cubics: Suppose $|z|>\max\{|b|,(|a|+2/s)^{1/2},(|a|+2/s')^{1/2}\}$ then $|z_n|\to\infty$ as $n\to\infty$. This gives an escape criterion for cubic polynomials

General Escape Criterion: Consider $|z|>\max\{|c|,(2/s)^{1/n},(2/s')^{1/n}\}$ then $|z_n|\to\infty$ as $n\to\infty$ is the escape criterion. (Escape Criterion derived in [3] & [19]).

Note that the initial value z_0 should be infinity, since infinity is the critical point of $z\to(z^n+c)^{-1}$. However instead of starting with z_0 = infinity, it is simpler to start with z_1 = c, which yields the same result. (*A critical point of* $z\to F(z)+c$ *is a point where* $F'(z)=0$ *)*. The role of critical points is explained in [1].

The purpose of this paper is to visualize the relative superior antifractals of the complex inverse function i.e., antifractals with respect to relative superior orbit and to analyze the pattern of symmetry among them.

3. GEOMETRY OF RELATIVE SUPERIOR MANDELBAR SETS

The results of plotting the Relative Superior Mandelbar set for the function A_c using Ishikawa Iterates, gives us the half moon shaped like crescent structure, hence, it can be named as Relative Superior Mandelbar set for quadratic. Crowe et. al [4], has considered it in formal analogy with Mandelbrot set and named it "Mandelbar set". The general escape criterion for higher powers of polynomials, $A_c(z)=z'^n+c$ where n is the degree of the polynomial, is given as $\max\{|c|,(2/s)^{1/n},(2/s')^{1/n}\}$ This can be used as the escape criterion for the function $A_c(z)$. We derive Relative Superior Mandelbar sets using this escape criteria. We have used the same escape criterion for generating the new Mandelbar sets for quadratic function for which the condition is $\max\{|c|, 2/s, 2/s'\}$.

The characteristics of the Relative Superior Julia set for a point inside the Relative Superior Mandelbar set can be given by observing the Relative Superior Mandelbar set. We know that, if *c* lies in A_c, the orbit of 0.0 does not escape to infinity. Hence we can say that if *c* does not lie in A_c then the Relative Superior Julia set J_c for Relative Superior Mandelbar set, is a Cantor set. The Relative Superior Julia set of A_c is either connected or totally disconnected, depending on, whether the orbit of 0 is bounded or escapes to infinity. We know that every Relative Superior Julia set is either:

· A Primary Relative Superior Julia set, or
· A Secondary Relative Superior Julia set

Primary Relative Superior Julia set are the Relative superior Julia set for the points attached to the main body of the Relative Superior Mandelbar set, whereas the name secondary Relative Superior Julia set can be given to those Julia set which belongs to the parts attached to the main body. We study here the primary Relative Superior Julia set for Relative Superior Mandelbar set (See Section 6). Further, we observe that the Relative Superior Julia set for Relative superior Mandelbar set consists of all c-values for which J_c is connected, or the orbit of 0.0 under $z'^2 + c$ does not tend to infinity.

We see that the Relative Superior Mandelbar set of quadratics consists of one crescent shaped body. Further, the Relative superior Mandelbar sets of higher polynomials contain the Main body having the number of parts attached to it less than that of Multicorns [4, 14, 15 &16].

Here, we are presenting the observation in the study of the Relative Superior Mandelbar sets from the figures mentioned in Section 5.

Relative Superior Mandelbar sets:

- Here we notice that the number of body parts in the Relative Superior Mandelbar sets is n-1, where n is the power of z'.
- As the value of s tend to 1 and s' tends to 1, the Relative Superior Mandelbar sets of order higher than two, have their main body get separated into n-1 equal parts which exists at some distance from each other.
- Starting with $A_c(z) = z'^n + c$, for n = 2,3,4,... and s < 1, s' < 1and applying the Ishikawa iterates we see that the Relative Superior Mandelbar sets of this function carries number of cuts in each crescent equals to n+1
- We also observe that for *n* is odd we have symmetry about both X and Y axis but for *n* is even the symmetry is maintained only along X axis.

Relative Superior Julia sets:

- Geometrical analysis of the Relative Superior Julia sets of inverse complex conjugate function shows that the boundary of the fixed point region forms a **(*n* - 1) crescent shaped petals like parts.**
- For each value of *c*, we can iterate the mapping and test whether the resulting sequence of z approaches a cycle or not. The points that lead to a cycle can be colored according to the length of the cycle and the points that never enter the cycle but wander chaotically are colored dark. ***The light color regions in the figures represent stable points while dark colored regions represent unstable points***.

- Relative Superior Julia sets of inverse complex conjugate function for quadratic function shows ball shaped figure maintaining symmetry along X axis. For cubic function Relative Superior Julia sets shows symmetry along X and Y axes both. Moreover this function also describes reflection and rotational symmetry. The biquadratic function shows us the fascinating results. Here we have ***central planet with satellite like structure*s** obtained that represents reflection as well as rotational symmetry.

4. FIXED POINTS

4.1 Fixed points of quadratic polynomial

Table 1: Orbit of F(z) for **s=1, s'=1** at
z_0= -0.7596358795+0.006005097i

Number of iteration *i*	|F(z)|	Number of iteration *i*	|F(z)|
1	0.75966	11	0.22073
2	0.44694	12	0.22071
3	0.37567	13	0.22067
4	0.37069	14	0.22069
5	0.16322	15	0.22068
6	0.22445	16	0.22068
7	0.22942	17	0.22068
8	0.21663	18	0.22068
9	0.22109	19	0.22068
10	0.22115	20	0.22068

Here we observe that the value converges to a fixed point after 15 iterations

Figure 1. Orbit of F(z) for **s=1, s'=1** at
z_0= -0.7596358795+0.006005097i

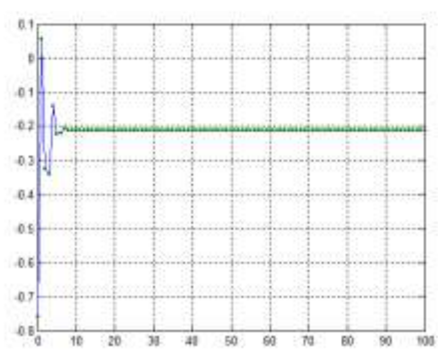


Table 2: Orbit of F(z) for **s=0.5, s'=0.7** at
z_0= -0.6160374839+0.0135629073i

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.61619	14	0.35866
2	0.5189	15	0.35835
3	0.288	16	0.35852
4	0.43079	17	0.35842
5	0.32218	18	0.35848
6	0.37886	19	0.35845
7	0.34703	20	0.35846
8	0.36492	21	0.35845
9	0.35484	22	0.35846
10	0.36049	23	0.35846
11	0.35732	24	0.35846
12	0.3591	25	0.35846
13	0.3581	26	0.35846

Here we observe that the value converges to a fixed point after 22 iterations

Figure 2. Orbit of F(z) for **s=0.5, s'=0.7** at
z_0= -0.6160374839+0.0135629073i

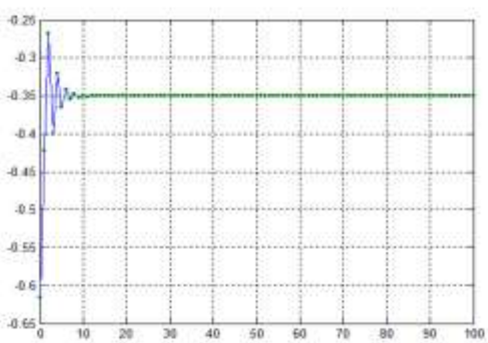
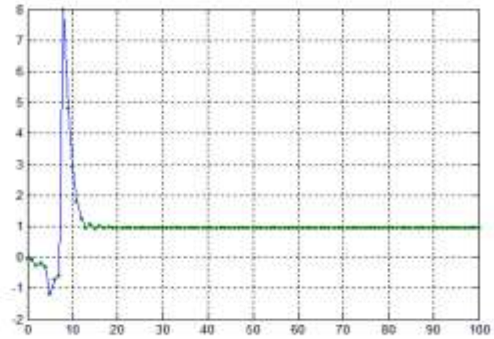


Table 3: Orbit of F(z) for **0.4, s'=0.1** at
z_0= -0.04164390139+0.006005097i

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.042075	16	0.93234
2	0.81996	17	1.001
3	0.2836	18	0.94791
4	1.101	19	0.96464
5	0.52569	20	0.95734
6	1.8414	21	0.95903
7	1.031	22	0.95897
8	0.74237	23	0.95875
9	8.0956	24	0.9589
10	4.8648	25	0.95884
11	2.9397	26	0.95885
12	1.8177	27	0.95885
13	1.2423	28	0.95885
14	0.97245	29	0.95885
15	1.0586	30	0.95885

Here the value converges to a fixed point after 26 iterations

Figure 3. : Orbit of F(z) for **0.4, s'=0.1** at
z_0= -0.04164390139+0.006005097i



4.2 Fixed points of Cubic polynomial

Table 1: Orbit of F(z) for **s=1, s'=1** at
z_0= 0.222879459+0.7693439369i

Number of iteration *i*	|F(z)|	Number of iteration *i*	|F(z)|
1	0.80098	6	0.13547
2	0.069844	7	0.13547
3	0.13321	8	0.13547
4	0.13536	9	0.13547
5	0.13546	10	0.13547

Here the value converges to a fixed point after 06 iterations

Figure 1 Orbit of F(z) for **s=1, s'=1** at
z_0= 0.222879459+0.7693439369i

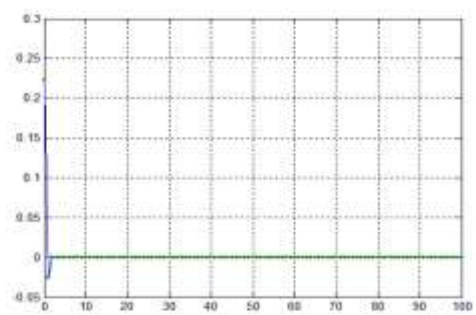


Table 2: Orbit of F(z) for **s=0.4,s'=0.2** at
z_0 = -0.0189704705+0.02867852789i

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
18	0.6098	28	0.50274
19	0.45643	29	0.50223
20	0.52977	30	0.50252
21	0.4871	31	0.50236
22	0.51134	32	0.50245
23	0.49733	33	0.5024
24	0.50536	34	0.50243
25	0.50073	35	0.50241
26	0.50339	36	0.50242
27	0.50186	37	0.50242

We skipped 17 iterations and value converges to a fixed point after 36 iterations

Figure 2 Orbit of F(z) for **s=0.4, s'=0.2** at **z_0 = -0.0189704705+0.02867852789i**

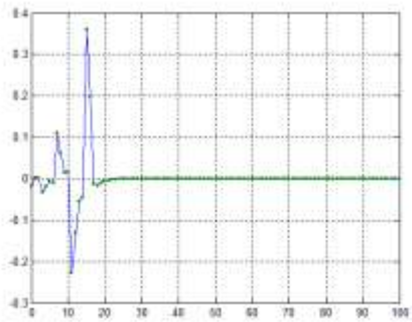
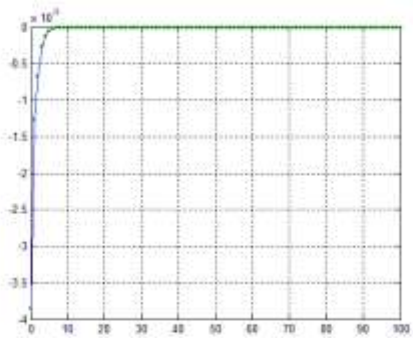


Table 3: Orbit of F(z) for **s=0.5, s'=0.7** at **z_0= -0.003854849909-0.01666833389i**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.017108	12	0.13646
2	0.2116	13	0.13602
3	0.096623	14	0.13627
4	0.1598	15	0.13613
5	0.12295	16	0.13621
6	0.1439	17	0.13617
7	0.13178	18	0.13619
8	0.13872	19	0.13618
9	0.13473	20	0.13619
10	0.13702	21	0.13618
11	0.1357	22	0.13618

Here the value converges to a fixed point after 21 iterations

Figure 3 Orbit of F(z) for **s=0.5, s'=0.7** at **z_0= -0.003854849909-0.01666833389i**



4.3 Fixed points of Bi-quadratic polynomial

Table 1: Orbit of F(z) for **s=1, s'=1** at **z_0= 0.4118247163+0.7164392648i**

Number of iteration *i*	|F(z)|	Number of iteration *i*	|F(z)|
1	0.82637	4	0.062471
2	0.0038329	5	0.062471
3	0.06247	6	0.062471

Here the value converges to a fixed point after 04 iterations

Figure 1 Orbit of F(z) for **s=1, s'=1** at **z_0= 0.4118247163+0.7164392648i**

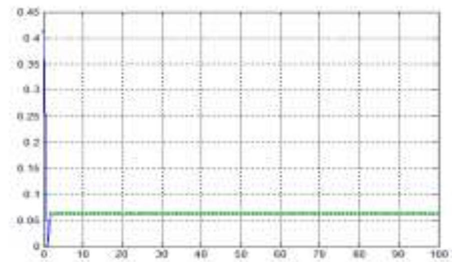


Table 2: Orbit of F(z) for **s=0.4, s'=0.2** at **z_0= -0.03408609109+0.04379414848i**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.055496	13	0.49886
2	0.76449	14	0.50069
3	0.31085	15	0.49959
4	0.61364	16	0.50025
5	0.43138	17	0.49986
6	0.54122	18	0.50009
7	0.47532	19	0.49995
8	0.51481	20	0.50003
9	0.49113	21	0.49998
10	0.50532	22	0.50001
11	0.49682	23	0.5
12	0.50191	24	0.5

Here the value converges to a fixed point after 23 iterations

Figure 2 Orbit of F(z) for **s=0.4, s'=0.2** at **z_0= -0.03408609109+0.04379414848i**

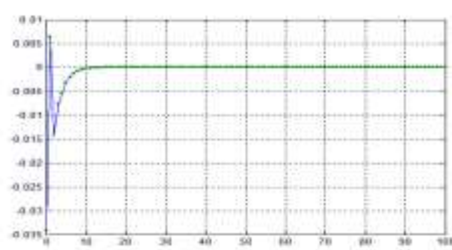
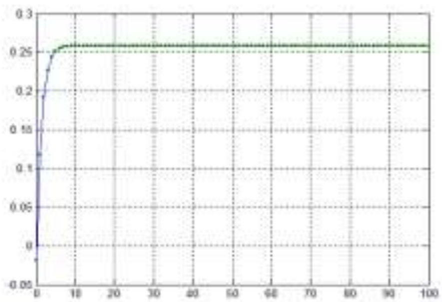


Table 3: Orbit of F(z) for **s=0.5, s'=0.7** at **z_0= -0.0189704705-0.001552713296i**

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1	0.019034	11	0.25855
2	0.11937	12	0.2586
3	0.19186	13	0.25863
4	0.22649	14	0.25865
5	0.24441	15	0.25865
6	0.252	16	0.25865
7	0.25581	17	0.25866
8	0.25733	18	0.25866
9	0.2581	19	0.25866
10	0.25839	20	0.25866

Here the value converges to a fixed point after 17 iterations

Figure 3. Orbit of F(z) for **s=0.5, s'=0.7** at **z_0= -0.0189704705-0.001552713296i**



5. GENERATION OF RELATIVE SUPERIOR MANDELBAR SETS

We generate Relative Superior Mandelbar sets for quadratic, cubic, biquadratic function and other higher order polynomials.

4.1 Relative Superior Mandelbar sets for Quadratic function:

Figure 1: Relative Superior Mandelbar set for s=s'=1

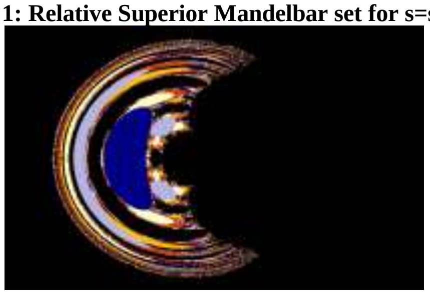


Figure 2: Relative Superior Mandelbar set for s=0.6, s'=0.2

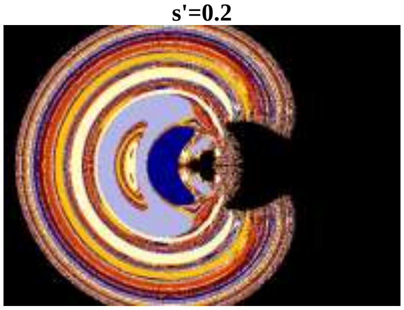
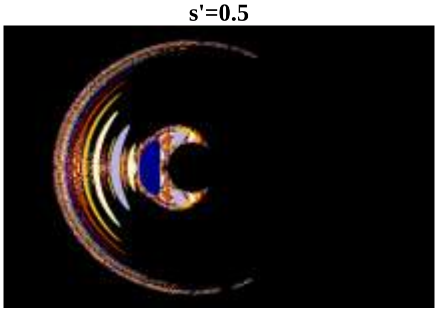


Figure 3: Relative Superior Mandelbar set for s=0.5, s'=0.5



4.2. Relative Superior Mandelbar set for Cubic function:

Figure 1: Relative Superior Mandelbar set for s=1, s'=1

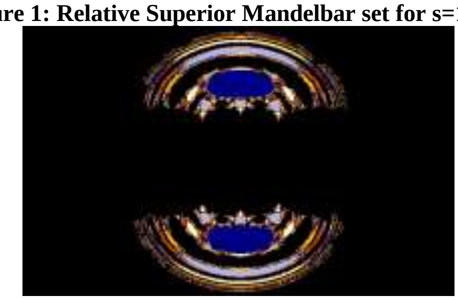


Figure 2: Relative Superior Mandelbar set for s=0.4, s'=0.2

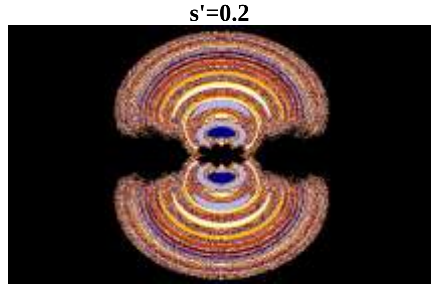
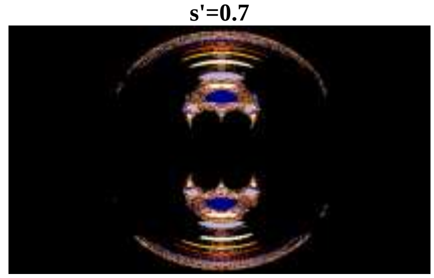


Figure 3: Relative Superior Mandelbar set for s=0.5, s'=0.7



4.3 Relative Superior Mandelbar set for Bi-quadratic function:

Figure 1: Relative Superior Mandelbar set for s=s'=1

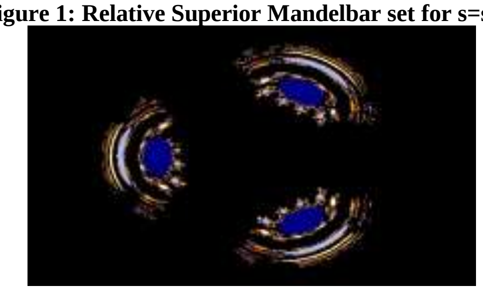


Figure 2: Relative Superior Mandelbar set for s=0.4 s'=0.2

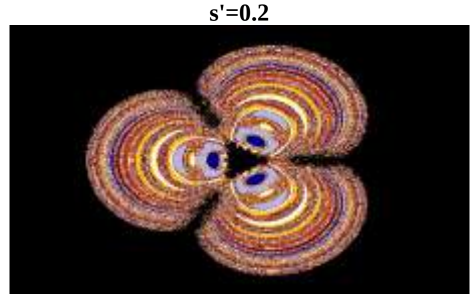
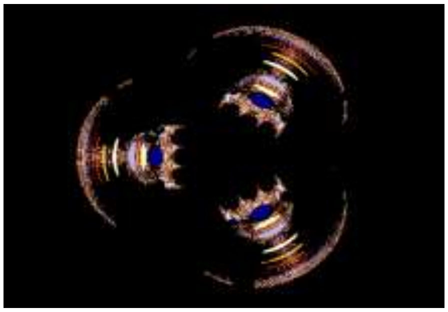


Figure 3: Relative Superior Mandelbar set for s=0.5, s'=0.7



4.4. Generalization of Relative Superior Mandelbar sets:

Figure 1: Relative Superior Mandelbar set for s=0.5, s'=0.3, n=15

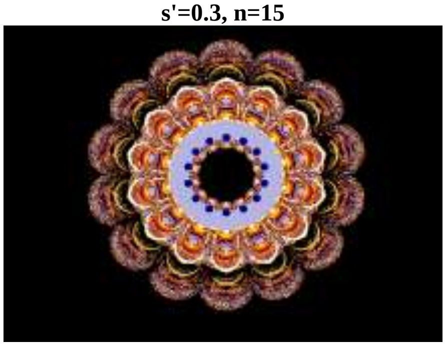


Figure 2: : Relative Superior Mandelbar set for s=0.5, s'=0.2, n=19

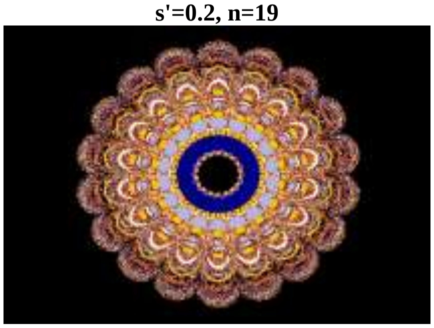
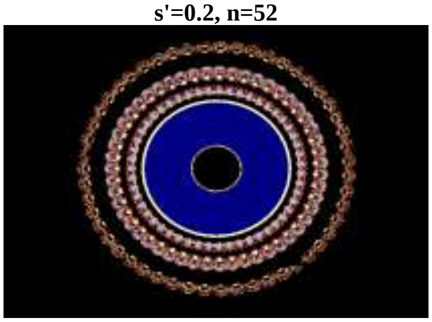


Figure 3: Relative Superior Multicorns for s=0.8, s'=0.2, n=52



6. GENERATION OF RELATIVE SUPERIOR JULIA SETS FOR MANDELBAR SETS

We present here some filled Relative Superior Julia sets for quadratic, cubic and biquadratic function.

6.1 Relative Superior Julia sets for Quadratic:

Figure 1: Relative Superior Julia Set for s=0.4, s'=0.1 c=-0.04164390139+0.006005097i

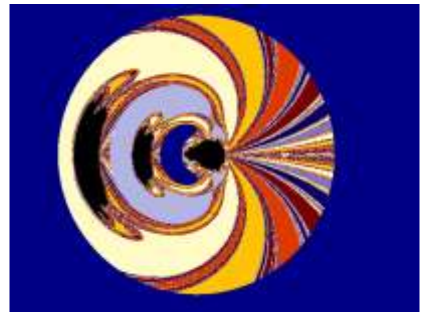
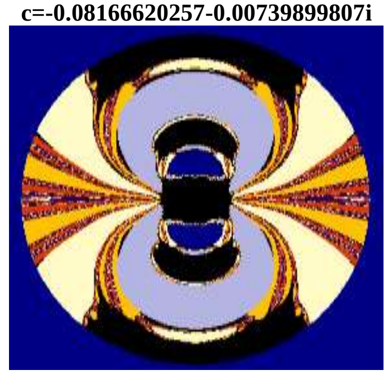


Figure 2: Relative Superior Julia Set for, s=0.6, s'=0.2 c=-0.08166620257-0.00739899807i



6.2 Relative Superior Julia sets for Cubic function:

Figure 1: Relative Superior Julia for s=0.4,s'=0.2 c=-0.0189704705+0.02867852789i

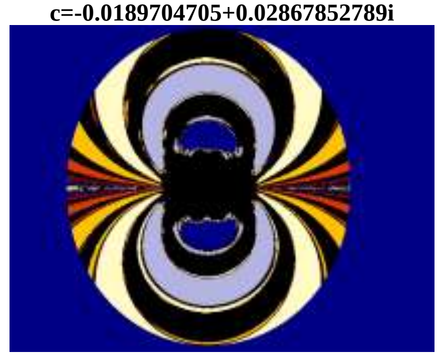
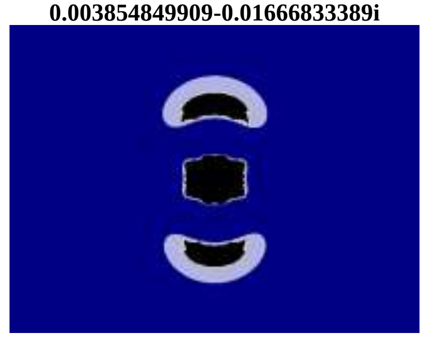


Figure 2: Relative Superior Julia Set for s=0.5,s'=0.7 c=-0.003854849909-0.01666833389i



6.3 Relative Superior Julia sets for Bi-quadratic function:

Figure 1: Relative Superior Julia for s=0.4, s'=0.2 c=-0.03408609109+0.04379414848i

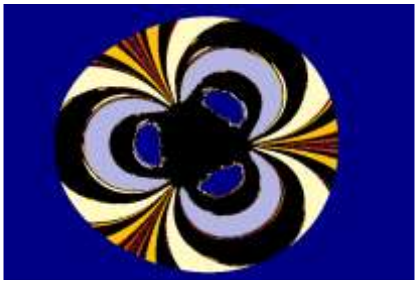
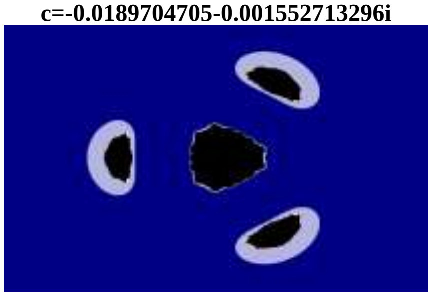


Figure 2: Relative Superior Julia for s=0.5, s'=0.7 c=-0.0189704705-0.001552713296i



7. CONCLUSION:

In the dynamics of antipolynomial of complex polynomial $z'^{n}+c$, where $n \geq 2$, there exist many Mandelbar sets for a value of *n* with respect to Relative Superior orbit. Further, for the odd values of *n*, all the Relative Superior Mandelbar sets are symmetrical objects, and for even values of *n*, all the Relative Superior Mandelbar sets are symmetrical about x-axis. Besides this, our antifractals are different from the normal Tricorns and Multicorns as they have (n-1) wings.

8. REFERENCES

[1] B. Branner, *The Mandelbrot Set*, Proceedings of Symposia in Applied Mathematics39 (1989), 75-105. Published as Chaos and Fractals: The Mathematics Behind the Computer Graphics, ed. R. L. Devaney, L. Keen.

[2] P. Blanchard, Complex Analytic Dynamics on the Riemann Sphere, Bulletin of the American Mathematical Society 11, 1 (1984), 85-141.

[3] Y. S. Chauhan, R. Rana and A. Negi, New Tricorns and Multicorns of Ishikawa Iterates, In Press, Int. Journal of Computer Application (Oct. 2010 Edition).

[4] W. D. Crowe, R. Hasson, P. J. Rippon, and P. E. D. Strain-Clark, “On the structure of the Mandelbar set”, Nonlinearity(2)(4)(1989), 541-553. MR1020441.

[5] S. Dhurandar, V. C. Bhavsar and U. G. Gujar, “Analysis of z-plane fractal images from $z \leftarrow z^{\alpha}+c$ for α < 0, Computers and Graphics 17, 1 (1993), 89-94.

[6] Robert L. Devaney, “A First Course in Chaotic Dynamical Systems: Theory and Experiment”, Addison-Wesley, 1992. MR1202237.

[7] U. G. Gujar and V. C. Bhavsar, Fractals from $z \leftarrow z^{\alpha}+c$ in the Complex c-Plane, Computers and Graphics 15, 3 (1991), 441-449.

[8] U. G. Gujar, V. C. Bhavsar and N. Vangala, Fractals from $z \leftarrow z^{\alpha}+c$ in the Complex z-Plane, Computers and Graphics 16, 1 (1992), 45-49.

[9] E. F. Glynn, The Evolution of the Gingerbread Man, Computers and Graphics 15,4 (1991), 579-582.

[10] S. Ishikawa, “Fixed points by a new iteration method”, Proc. Amer. Math. Soc.44 (1974), 147-150.

[11] Eike Lau and Dierk Schleicher, “Symmetries of fractals revisited.”, Math. Intelligencer (18)(1)(1996), 45-51.

[12] B. B. Mandelbrot, The Fractal Geometry of Nature, W. H. Freeman, New York,1983.

[13] J. Milnor, “Dynamics in one complex variable; Introductory lectures”, Vieweg (1999).

[14] N. Shizuo and Dierk Schleicher, “Non-local connectivity of the tricorn and multicorns”, Dynamical systems and chaos (1) (Hachioji, 1994), 200-203, World Sci. Publ., River Edge, NJ, 1995. MR1479931.

[15] N. Shizuo and Dierk Schleicher, “On multicorns and unicorns: I. Antiholomorphic dynamics. hyperbolic components and real cubic polynomials”, Internat. J. Bifur. Chaos Appl. Sci. Engrg, (13)(10)(2003), 2825-2844.

[16] Ashish Negi, “Generation of Fractals and Applications”, Thesis, Gurukul Kangri Vishwvidyalaya, (2005).

[17] H. Peitgen and P. H. Richter, The Beauty of Fractals, Springer-Verlag, Berlin,1986.

[18] H. Peitgen, Jurgens and Saupe, “Chaos and Fractals, Springer-Verlag”, NewYork, Inc., 1992.

[19] R. Rana, Y. S. Chauhan and A. Negi, Non Linear dynamics of Ishikawa Iteration, In Press, Int. Journal of Computer Application (Oct. 2010 Edition).

[20] K. W. Shirriff, “An investigation of fractals generated by $z \to z^{-n}+c$”, Computers and Graphics 13, 4 (1993), 603-607.

[21] R. Winters, “Bifurcations in families of Antiholomorphic and biquadratic maps”, Thesis, Boston Univ. (1990).